Conservation Laws Related to the Kac-Moody-Virasoro Structure of the Potential Nizhnik-Novikov-Veselov Equation

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We derive the symmetry group by the standard Lie symmetry method and prove it constitutes to the Kac-Moody-Virasoro algebra. Then we construct the conservation laws corresponding to the Kac-Moody-Virasoro symmetry algebra up to second-order group invariants.

Key words: Potential Nizhnik-Novikov-Veselov Equation; Lie Point Symmetry Group; Conservation Laws.

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1. Introduction

It is of fundamental importance to obtain conservation laws of a given nonlinear system. As for the soliton theory, conservation laws facilitate the study of qualitative properties of partial differential equations (PDEs), such as bi- or tri-Hamiltonian structures. They often guide the choice of solution methods or reveal the nature of special solutions. For example, the existence of a large number of conservation laws is a predictor for complete integrability of the PDE [1], i. e. solvability by the inverse scattering transform [1] and the existence of solitons [2]. Conserved densities also aid in the design of numerical solvers for PDEs [3].

There are various methods to compute conservation laws of nonlinear PDEs. A common approach relies on the link between conservation laws and symmetries as stated in Noether's theorem [4 – 6]. In the classical Noether theorem [7] it is valid that if a given system of differential equations has a variational principle, then a continuous symmetry (point, contactor higher order) that leaves the action functional within a divergence invariant yields a conservation law [8 – 11]. In the last few years, effective methods have been devised for finding conservation laws for the very special class of so-called Lax equations. In 2003, Anco and Bluman gave the multiplier method for finding the local conservation laws of PDEs [12, 13]. In 2000, Kara and Mahomed [14] presented the direct relationship between the conserved vector of a PDE and the Lie-Bäcklund symmetry generators of the PDE, from which it is possible for us to obtain conservation laws from symmetries (see, e. g., [15]). In the new method of Kara and Mahomed, one can use any Lie-Bäcklund operator of a PDE system to generate conservation laws without converting the operator to a canonical one, so a point symmetry generator remains of point type. Another advantage of this approach is that for point symmetry the order of the generated conservation Law remains the same as for the original one remains the same. Furthermore, by using this method one can prove that the Lie algebra of Lie-Bäcklund symmetry generators of the conserved form is a subalgebra of the symmetries of the system itself. In this letter, we discuss the conservation laws relating to the Kac-Moody-Virasoro algebra (KMVA) of the Nizhnik-Novikov-Veselov equation (NNVE) [16 – 19] using the Lie-Bäcklund operator.

The Nizhnik-Novikov-Veselov equation

$$v_t + v_{xxx} + v_{yyy} + 3(v\partial_y^{-1}v_x)_x + 3(v\partial_x^{-1}v_y)_y = 0, \quad (1)$$

wherein x and y occur in a symmetric manner, is the famous extension of the Kortheweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0. \quad (2)$$

Various integrable properties such as the inverse scattering transformation, Bäcklund transformation, the soliton-like solutions, and symmetry algebra of (1) are discussed by various authors [20 – 24]. We would like to derive the conservation laws related to the symmetries and algebras for the NNVE (1) in the potential form ($v = u_{xy}$)

$$u_{txy} + u_{xxxxy} + u_{xyyyy} + 3(u_{xy}u_{xx})_x + 3(u_{xy}u_{yy})_y = 0. \quad (3)$$

This paper is organized as follows. In Section 2, we derive the symmetry group to the potential NNVE (PNNVE) by using the classical Lie method and prove the constitute infinite dimensional Kac-Moody-Virasoro symmetry algebra [25]. In Section 3, we first review some basic notions about Lie-Bäcklund operators and then use them. We finally obtain conservation laws related to the infinite dimensional Kac-Moody-Virasoro symmetry algebra as PNNVE possesses up to second-order group invariants. It is emphasized that equations with the same symmetries may possess the same types of conservation laws. The last section is a short summary and discussion.

2. Lie Point Symmetries and Kac-Moody-Virasoro Structure of the Potential Nizhnik-Novikov-Veselov Equation

To study the symmetry of (3), we search for the Lie point symmetry transformations in the vector form

$$V = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u},$$

where X, Y, T, and U are functions with respect to x, y, t, u, which means that (3) is invariant under the point transformation

$$\{x, y, t, u\} \to \{x + \varepsilon X, y + \varepsilon Y, t + \varepsilon T, u + \varepsilon U\}$$

with infinitesimal parameter ε.

In other words, the symmetry of (3) can be written in the function form

$$\sigma = Xu_x + Yu_y + Tu_t - U, \tag{4}$$

where the symmetry σ is a solution of the linearized form of (3),

$$\begin{aligned}&\sigma_{xyt} + \sigma_{xxxxy} + \sigma_{xyyyy} + 3u_{xxy}\sigma_{xx} + 3\sigma_{xxy}u_{xx}\\ &\quad + 3u_{xy}\sigma_{xxx} + 3\sigma_{xy}u_{xxx} + 3u_{xyy}\sigma_{yy} + 3\sigma_{xyy}u_{yy}\\ &\quad + 3u_{xy}\sigma_{yyy} + 3\sigma_{xy}u_{yyy} = 0,\end{aligned} \tag{5}$$

which is obtained by substituting $u = u + \varepsilon\sigma$ into (3) and dropping the nonlinear terms in σ.

It is easy to determine $X(x,y,t,u)$, $Y(x,y,t,u)$, $T(x,y,t,u)$, and $U(x,y,t,u)$ by substituting (4) into (5) and eliminating u_{xxxxy} and its higher-order derivatives by means of the PNNVE (3). The results are:

$$X(x,y,t,u) = \frac{1}{3}T_t x + X, \tag{6}$$

$$Y(x,y,t,u) = \frac{1}{3}T_t y + Y, \tag{7}$$

$$T(x,y,t,u) = T, \tag{8}$$

$$\begin{aligned}U(x,y,t,u) = {}&\frac{1}{54}(x^3 + y^3)T_{tt} + \frac{1}{6}Y_t y^2 + \frac{1}{6}X_t x^2\\ &+ Z_1 x + Z_2 y + Z_3,\end{aligned} \tag{9}$$

where X, Y, T, Z_1, Z_2, and Z_3 are arbitrary functions of t.

The vector form of the Lie point symmetries reads

$$\begin{aligned}V = {}&\left\{\frac{1}{3}T_t(t)x\frac{\partial}{\partial x} + \frac{1}{3}T_t y\frac{\partial}{\partial y} + T\frac{\partial}{\partial t}\right.\\ &\left.-\frac{1}{54}(T_{tt}x^3 + T_{tt}y^3)\frac{\partial}{\partial u}\right\}\\ &+\left\{Y\frac{\partial}{\partial y} - \frac{1}{6}Y_t y^2\frac{\partial}{\partial u}\right\}\\ &+\left\{X\frac{\partial}{\partial x} - \frac{1}{6}X_t x^2\frac{\partial}{\partial u}\right\} + \left\{-Z_1 x\frac{\partial}{\partial u}\right\}\\ &+\left\{-Z_2 y\frac{\partial}{\partial u}\right\} + \left\{-Z_3\frac{\partial}{\partial u}\right\}\\ \equiv {}& V_1(T(t)) + V_2(Y(t)) + V_3(X(t))\\ &+ V_4(Z_1(t)) + V_5(Z_2(t)) + V_6(Z_3(t)).\end{aligned} \tag{10}$$

It is easy to verify that the symmetries V_i, $i = 1, 2, 3, 4, 5, 6$, constitute an infinite dimensional Kac-Moody-Virasoro [25] type symmetry algebra $\boldsymbol{S}$ with the following non-zero commutation relations:

$$[V_1(T),\ V_6(Z_3)] = V_6(TZ_{3t}), \tag{11}$$

$$[V_2(Y),\ 5(Z_2)] = V_6(YZ_2), \tag{12}$$

$$[V_1(T),\ V_5(Z_2)] = V_5(TZ_{2t} + \frac{1}{3}T_t Z_2), \tag{13}$$

$$[V_4(Z_1),\ V_1(T)] = V_4\left(-\frac{1}{3}T_t Z_1 - TZ_{1t}\right), \tag{14}$$

$$[V_3(X),\ V_1(T)] = V_3\left(-TX_t + \frac{1}{3}XT_t\right), \tag{15}$$

$$[V_2(Y),\ V_1(T)] = V_2\left(\frac{1}{3}YT_t - Y_t T\right), \tag{16}$$

$$[V_1(T_1),\ V_1(T_2)] = V_1(T_1 T_{2t} - T_2 T_{1t}), \tag{17}$$

$$[V_2(Y_1),\ V_2(Y_2)] = V_5\left(\frac{1}{3}(Y_1 Y_{2t} - Y_2 Y_{1t})\right), \tag{18}$$

$$[V_4(Z_1),\ V_3(X)] = V_6(-XZ_1), \tag{19}$$

$$[V_3(X_1),\ V_3(X_2)] = V_4\left(\frac{1}{3}(X_1X_{2t} - X_2X_{1t})\right). \tag{20}$$

It should be emphasized that the algebra is infinite dimensional because the generators V_1, V_2, V_3, V_4, V_5, and V_6 all contain *arbitrary* functions. The algebra is closed because all the commutators can be expressed by the generators belonging to the generator set usually with different functions, and the generators contained *different* functions belonging to the set. Especially, it is clear that the symmetry $V_1(T)$ constitute an centerless Virasoro symmetry algebra.

3. Conservation Laws Related to the KMVA of PNNVE

In order to obtain conservation laws related to the symmetry (10), we need some basic notions about Lie-Bäcklund operators first.

A Lie-Bäcklund operator is given by

$$X_0 = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \zeta_i \frac{\partial}{\partial u_i} + \zeta_{i_1 i_2} \frac{\partial}{\partial u_{i_1 i_2}} + \cdots, \tag{21}$$

where ξ^i, η and the additional coefficients are

$$\begin{aligned} \zeta_i &= \mathrm{D}_i(W) + \xi^j u_{ij}, \\ \zeta_{i_1 i_2} &= \mathrm{D}_{i_1 i_2}(W) + \xi^j u_{j i_1 i_2}. \end{aligned} \tag{22}$$

W is the Lie characteristic function defined by

$$W = \eta - \xi^j u_j \tag{23}$$

with D_i being the operator of total differentiation,

$$\begin{aligned} &\mathrm{D}_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots, \\ &i = 1, \cdots, n, \end{aligned} \tag{24}$$

as

$$u_i = \mathrm{D}_i(u), \quad u_{ij} = \mathrm{D}_j \mathrm{D}_i(u). \tag{25}$$

These definitions and results related to the Lie-Bäcklund operator can be found in [26] and the repeated indices mean the summations according to the Einstein summation rule.

Correspondingly, the second-order Lie-Bäcklund operator of the vector field V defined by (10) is given by

$$\begin{aligned} X_0 = {} & \xi^x \frac{\partial}{\partial x} + \xi^y \frac{\partial}{\partial y} + \xi^t \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} + \zeta_x \frac{\partial}{\partial u_x} \\ & + \zeta_y \frac{\partial}{\partial u_y} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_u \frac{\partial}{\partial u_{xx}} + \zeta_{xy} \frac{\partial}{\partial u_{xy}} \\ & + \zeta_{xt} \frac{\partial}{\partial u_{xt}} + \zeta_{yy} \frac{\partial}{\partial u_{yy}} + \zeta_{yt} \frac{\partial}{\partial u_{yt}} + \zeta_{tt} \frac{\partial}{\partial u_{tt}}, \end{aligned} \tag{26}$$

where the coefficients ξ^x, ξ^y, $\cdots$, ζ_{tt} can be derived straightforwardly from (21) – (25).

Theorem ([14, 27]): Suppose that X_0 is a Lie-Bäcklund symmetry of (3) such that the conservation vector $T' = (T_1,\ T_2,\ T_3)$ is invariant under X_0. Then

$$\begin{aligned} &X_0(T_i) + \sum_{j=1}^{3} T_i \mathrm{D}_j \xi_j - \sum_{j=1}^{3} T_j \mathrm{D}_j(\xi_i) = 0, \\ &i = 1,\ 2,\ 3, \end{aligned} \tag{27}$$

where $\mathrm{D}_1 = \mathrm{D}_x$, $\mathrm{D}_2 = \mathrm{D}_y$, $\mathrm{D}_3 = \mathrm{D}_t$, and ξ_i are determined by (26).

A Lie-Bäcklund symmetry X_0 is said to be associated with a conserved vector T' of (3) if X_0 and T' satisfy relations (27).

Now, we construct the corresponding conservation laws relating to (10) in the form

$$\mathrm{D}_x J_1 + \mathrm{D}_y J_2 + \mathrm{D}_t \rho = 0, \tag{28}$$

where $T_1 = J_1$, $T_2 = J_2$, $T_3 = \rho$ with J_1, J_2, and ρ being functions of $\{x,\ y,\ t,\ u,\ u_x,\ u_y,\ \cdots,\ u_{tt}\}$.

In terms of $T' = (J_1,\ J_2,\ \rho)$, (27) is equivalent to the following three equations:

$$\begin{aligned} &\left(\frac{1}{3}xT_t + X\right)\frac{\partial J_1}{\partial x} + \left(\frac{1}{3}yT_t + Y\right)\frac{\partial J_1}{\partial y} + T\frac{\partial J_1}{\partial t} \\ &+ \left(\frac{1}{54}(x^3+y^3)T_{tt} + \frac{1}{6}Y_t y^2 + \frac{1}{6}X_t x^2 + Z_1 x + Z_2 y + Z_3\right) \\ &\cdot \frac{\partial J_1}{\partial u} + \left(\frac{1}{18}x^2 T_{tt} + \frac{1}{3}X_t x + Z_1 - \frac{1}{3}T_t u_x\right)\frac{\partial J_1}{\partial u_x} \\ &+ \left(\frac{1}{18}y^2 T_{tt} + \frac{1}{3}Y_t y + Z_2 - \frac{1}{3}T_t u_y\right)\frac{\partial J_1}{\partial u_y} \\ &+ \left[\frac{1}{54}(x^3+y^3)T_{ttt} + \frac{1}{6}Y_{tt}y^2 + \frac{1}{6}X_{tt}x^2 + Z_{1t}x + Z_{2t}y + Z_{3t}\right. \\ &\left. - \left(\frac{1}{3}T_{tt}x + X_t\right)u_x - \left(\frac{1}{3}T_{tt}y + Y_t\right)u_y - T_t u_t\right]\frac{\partial J_1}{\partial u_t} \\ &+ \left(\frac{1}{9}T_{tt}x + \frac{1}{3}X_t - \frac{2}{3}T_t u_{xx}\right)\frac{\partial J_1}{\partial u_{xx}} - \frac{2}{3}T_t u_{xy}\frac{\partial J_1}{\partial u_{xy}} \end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{1}{18}x^2T_{ttt}+\frac{1}{3}X_{tt}x+Z_{1t}-\frac{1}{3}T_{tt}u_x-\frac{4}{3}T_tu_{tx}-\left(\frac{1}{3}T_{tt}x+X_t\right)u_{xx}-\left(\frac{1}{3}T_{tt}y+Y_t\right)u_{xy}\right]\frac{\partial J_1}{\partial u_{xt}}\\
&+\left(\frac{1}{9}T_{tt}y+\frac{1}{3}Y_t-\frac{2}{3}T_tu_{yy}\right)\frac{\partial J_1}{\partial u_{yy}}+\left[\frac{1}{18}y^2T_{ttt}+\frac{1}{3}Y_{tt}y+Z_{2t}-\frac{1}{3}T_{tt}u_y-\frac{4}{3}T_tu_{yt}-\left(\frac{1}{3}T_{tt}x+X_t\right)u_{xy}\right.\\
&\left.-\left(\frac{1}{3}T_{tt}y+Y_t\right)u_{yy}\right]\frac{\partial J_1}{\partial u_{yt}}+\left[\frac{1}{54}(x^3+y^3)T_{tttt}+\frac{1}{6}Y_{ttt}y^2+\frac{1}{6}X_{ttt}x^2+Z_{1tt}x+Z_{2tt}y+Z_{3tt}-\left(\frac{1}{3}T_{ttt}x+X_{tt}\right)u_x\right.\\
&\left.-2\left(\frac{1}{3}T_{tt}x+X_t\right)u_{xt}-\left(\frac{1}{3}T_{ttt}y+Y_{tt}\right)u_y-2\left(\frac{1}{3}T_{tt}y+Y_t\right)u_{yt}-T_{tt}u_t-2T_tu_{tt}\right]\frac{\partial J_1}{\partial u_{tt}}+\frac{4}{3}T_tJ_1-\left(\frac{1}{3}T_{tt}x+X_t\right)\rho=0,
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&\left(\frac{1}{3}xT_t+X\right)\frac{\partial J_2}{\partial x}+\left(\frac{1}{3}yT_t+Y\right)\frac{\partial J_2}{\partial y}+T\frac{\partial J_2}{\partial t}+\left(\frac{1}{54}(x^3+y^3)T_{tt}+\frac{1}{6}Y_ty^2+\frac{1}{6}X_tx^2+Z_1x+Z_2y+Z_3\right)\frac{\partial J_2}{\partial u}\\
&+\left(\frac{1}{18}x^2T_{tt}+\frac{1}{3}X_tx+Z_1-\frac{1}{3}T_tu_x\right)\frac{\partial J_2}{\partial u_x}+\left(\frac{1}{18}y^2T_{tt}+\frac{1}{3}Y_ty+Z_2-\frac{1}{3}T_tu_y\right)\frac{\partial J_2}{\partial u_y}+\left[\frac{1}{54}(x^3+y^3)T_{ttt}+\frac{1}{6}Y_{tt}y^2\right.\\
&\left.+\frac{1}{6}X_{tt}x^2+Z_{1t}x+Z_{2t}y+Z_{3t}-\left(\frac{1}{3}T_{tt}x+X_t\right)u_x-\left(\frac{1}{3}T_{tt}y+Y_t\right)u_y-T_tu_t\right]\frac{\partial J_2}{\partial u_t}+\left(\frac{1}{9}T_{tt}x+\frac{1}{3}X_t-\frac{2}{3}T_tu_{xx}\right)\frac{\partial J_2}{\partial u_{xx}}\\
&-\frac{2}{3}T_tu_{xy}\frac{\partial J_2}{\partial u_{xy}}+\left[\frac{1}{18}x^2T_{ttt}+\frac{1}{3}X_{tt}x+Z_{1t}-\frac{1}{3}T_{tt}u_x-\frac{4}{3}T_tu_{tx}-\left(\frac{1}{3}T_{tt}x+X_t\right)u_{xx}-\left(\frac{1}{3}T_{tt}y+Y_t\right)u_{xy}\right]\frac{\partial J_2}{\partial u_{xt}}\\
&+\left(\frac{1}{9}T_{tt}y+\frac{1}{3}Y_t-\frac{2}{3}T_tu_{yy}\right)\frac{\partial J_2}{\partial u_{yy}}+\left[\frac{1}{18}y^2T_{ttt}+\frac{1}{3}Y_{tt}y+Z_{2t}-\frac{1}{3}T_{tt}u_y-\frac{4}{3}T_tu_{yt}-\left(\frac{1}{3}T_{tt}x+X_t\right)u_{xy}\right.\\
&\left.-\left(\frac{1}{3}T_{tt}y+Y_t\right)u_{yy}\right]\frac{\partial J_2}{\partial u_{yt}}+\left[\frac{1}{54}(x^3+y^3)T_{tttt}+\frac{1}{6}Y_{ttt}y^2+\frac{1}{6}X_{ttt}x^2+Z_{1tt}x+Z_{2tt}y+Z_{3tt}-\left(\frac{1}{3}T_{ttt}x+X_{tt}\right)u_x\right.\\
&\left.-2\left(\frac{1}{3}T_{tt}x+X_t\right)u_{xt}-\left(\frac{1}{3}T_{ttt}y+Y_{tt}\right)u_y-2\left(\frac{1}{3}T_{tt}y+Y_t\right)u_{yt}-T_{tt}u_t-2T_tu_{tt}\right]\frac{\partial J_2}{\partial u_{tt}}+\frac{4}{3}T_tJ_2-\left(\frac{1}{3}T_{tt}y+Y_t\right)\rho=0,
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&\left(\frac{1}{3}xT_t+X\right)\frac{\partial \rho}{\partial x}+\left(\frac{1}{3}yT_t+Y\right)\frac{\partial \rho}{\partial y}+T\frac{\partial \rho}{\partial t}+\left(\frac{1}{54}(x^3+y^3)T_{tt}+\frac{1}{6}Y_ty^2+\frac{1}{6}X_tx^2+Z_1x+Z_2y+Z_3\right)\frac{\partial \rho}{\partial u}\\
&+\left(\frac{1}{18}x^2T_{tt}+\frac{1}{3}X_tx+Z_1-\frac{1}{3}T_tu_x\right)\frac{\partial \rho}{\partial u_x}+\left(\frac{1}{18}y^2T_{tt}+\frac{1}{3}Y_ty+Z_2-\frac{1}{3}T_tu_y\right)\frac{\partial \rho}{\partial u_y}+\left[\frac{1}{54}(x^3+y^3)T_{ttt}+\frac{1}{6}Y_{tt}y^2\right.\\
&\left.+\frac{1}{6}X_{tt}x^2+Z_{1t}x+Z_{2t}y+Z_{3t}-\left(\frac{1}{3}T_{tt}x+X_t\right)u_x-\left(\frac{1}{3}T_{tt}y+Y_t\right)u_y-T_tu_t\right]\frac{\partial \rho}{\partial u_t}+\left(\frac{1}{9}T_{tt}x+\frac{1}{3}X_t-\frac{2}{3}T_tu_{xx}\right)\frac{\partial \rho}{\partial u_{xx}}\\
&-\frac{2}{3}T_tu_{xy}\frac{\partial \rho}{\partial u_{xy}}+\left[\frac{1}{18}x^2T_{ttt}+\frac{1}{3}X_{tt}x+Z_{1t}-\frac{1}{3}T_{tt}u_x-\frac{4}{3}T_tu_{tx}-\left(\frac{1}{3}T_{tt}x+X_t\right)u_{xx}-\left(\frac{1}{3}T_{tt}y+Y_t\right)u_{xy}\right]\frac{\partial \rho}{\partial u_{xt}}\\
&+\left(\frac{1}{9}T_{tt}y+\frac{1}{3}Y_t-\frac{2}{3}T_tu_{yy}\right)\frac{\partial \rho}{\partial u_{yy}}+\left[\frac{1}{18}y^2T_{ttt}+\frac{1}{3}Y_{tt}y+Z_{2t}-\frac{1}{3}T_{tt}u_y-\frac{4}{3}T_tu_{yt}-\left(\frac{1}{3}T_{tt}x+X_t\right)u_{xy}\right.\\
&\left.-\left(\frac{1}{3}T_{tt}y+Y_t\right)u_{yy}\right]\frac{\partial \rho}{\partial u_{yt}}+\left[\frac{1}{54}(x^3+y^3)T_{tttt}+\frac{1}{6}Y_{ttt}y^2+\frac{1}{6}X_{ttt}x^2+Z_{1tt}x+Z_{2tt}y+Z_{3tt}-\left(\frac{1}{3}T_{ttt}x+X_{tt}\right)u_x\right.\\
&\left.-2\left(\frac{1}{3}T_{tt}x+X_t\right)u_{xt}-\left(\frac{1}{3}T_{ttt}y+Y_{tt}\right)u_y-2\left(\frac{1}{3}T_{tt}y+Y_t\right)u_{yt}-T_{tt}u_t-2T_tu_{tt}\right]\frac{\partial \rho}{\partial u_{tt}}+\frac{2}{3}T_t\rho=0.
\end{aligned}
\tag{31}
$$

The solutions J_1, J_2, and ρ of (29) – (31) can be directly solved:

$$\rho=f_0(t)K_1(t_1,\ t_2,\ t_3,\cdots,\ t_{12}), \tag{32}$$

$$
\begin{aligned}
J_2=&[f_1(t)+f_2(t)y]K_1(t_1,\ t_2,\ t_3,\cdots,t_{12})\\
&+f_3(t)K_2(t_1,\ t_2,\ t_3,\ \cdots,\ t_{12}),
\end{aligned}
\tag{33}
$$

$$J_1 = [f_4(t) + f_5(t)x]K_1(t_1,\, t_2,\, t_3, \cdots, t_{12}) + f_6(t)K_3(t_1,\, t_2,\, t_3,\, \cdots,\, t_{12}), \tag{34}$$

where K_1, K_2, and K_3 are arbitrary functions of $\{t_1,\, t_2,\, t_3,\, \cdots,\, t_{12}\}$, and f_i, $i = 0,\, 1,\, \cdots,\, 10$ are functions fixed by:

$$f_0 = T^{-\frac{2}{3}}, \tag{35}$$

$$f_1 = YT^{-\frac{5}{3}}, \quad f_2 = \frac{1}{3}T_tT^{-\frac{5}{3}}, \quad f_3 = T^{-\frac{4}{3}}, \tag{36}$$

$$f_4 = X_{1t}T^{-\frac{1}{3}}, \quad f_5 = \frac{1}{3}T_tT^{-\frac{5}{3}}, \quad f_6 = T^{-\frac{4}{3}}, \tag{37}$$

with the invariants being

$$t_1 = (x - T^{\frac{1}{3}}X_1)T^{-\frac{1}{3}}, \tag{38}$$

$$t_2 = (y - Y_1T^{\frac{1}{3}})T^{-\frac{1}{3}}, \tag{39}$$

$$\begin{aligned} t_3 = \frac{1}{54}T^{-\frac{4}{3}}\Big(&54uT^{\frac{4}{3}} - T_tT^{\frac{1}{3}}y^3 - T_tx^3T^{\frac{1}{3}} \\ &-9XT^{\frac{1}{3}}x^2 - 9Yy^2T^{\frac{1}{3}} - 54yY_2T + 18yY_3T \\ &-54Y_4T^{\frac{4}{3}} + 54Y_5T^{\frac{4}{3}} - 18Y_6T^{\frac{4}{3}} \\ &+54Y_7T^{\frac{4}{3}} - 18Y_8T^{\frac{4}{3}} - 54xY_9T + 18xY_{10}T\Big), \end{aligned} \tag{40}$$

$$\begin{aligned} t_4 = \frac{1}{18}T^{-\frac{2}{3}}\big(&18Tu_x - 6Xx - T_tx^2 \\ &-18Y_{10}T^{\frac{2}{3}} + 6Y_{11}T^{\frac{2}{3}}\big), \end{aligned} \tag{41}$$

$$\begin{aligned} t_5 = -\frac{1}{18}T^{-\frac{2}{3}}\big(&-18u_yT + T_ty^2 + 6Yy \\ &+18Y_2T^{\frac{2}{3}} - 6Y_3T^{\frac{2}{3}}\big), \end{aligned} \tag{42}$$

$$\begin{aligned} t_6 = {}&u_tT + u_yY + \frac{1}{3}u_yT_ty + Xu_x + \frac{1}{3}T_txu_x \\ &- \frac{1}{54}T_{tt}y^3 - \frac{1}{54}T_{tt}x^3 - \frac{1}{6}X_tx^2 - \frac{1}{6}Y_ty^2 \\ &- Z_2y - Z_1x - Z_3, \end{aligned} \tag{43}$$

$$t_7 = -\frac{1}{9}T^{-\frac{1}{3}}(-9u_{xx}T + 3X + T_tx), \tag{44}$$

$$t_8 = u_{xy}T^{\frac{2}{3}}, \tag{45}$$

$$\begin{aligned} t_9 = -\frac{1}{18}T^{\frac{1}{3}}(&-18u_{xt}T - 18u_{xy}Y - 6u_{xy}yT_t \\ &- 18u_{xx}X - 6u_{xx}xT_t + 6X_tx + 18Z_1 \\ &- 6T_tu_x + x^2T_{tt}), \end{aligned} \tag{46}$$

$$t_{10} = -\frac{1}{9}T^{-\frac{1}{3}}(-9Tu_{yy} + 3Y + T_ty), \tag{47}$$

$$\begin{aligned} t_{11} = -\frac{1}{18}T^{\frac{1}{3}}(&-18Tu_{yt} - 6xu_{xy}T_t - 18Yu_{yy} \\ &- 6T_tu_{yy}y - 18Xu_{xy} + 18Z_2 - 6T_tu_y \\ &+ y^2T_{tt} + 6Y_ty), \end{aligned} \tag{48}$$

$$\begin{aligned} t_{12} = {}&\frac{1}{3}T_tXu_x - \frac{1}{9}T_tx^2X_t - \frac{1}{3}XxX_t + TX_tu_x - Z_2Y \\ &-Z_{3t}T + \frac{1}{3}TT_{tt}xu_x + \frac{2}{3}TxT_tu_{xt} + \frac{2}{3}Tyu_{yt}T_t + \frac{2}{3}XT_tyu_{xy} \\ &+ \frac{2}{3}u_{xx}xXT_t + \frac{2}{3}u_{yy}yYT_t + \frac{2}{9}xyT_t^2u_{xy} + \frac{2}{3}u_{xy}xYT_t \\ &+ \frac{1}{3}Tu_{yy}yT_{tt} + Tu_tT_t + Tu_yY_t - TyZ_{2t} - TxZ_{1t} \\ &- \frac{1}{18}y^2YT_{tt} - \frac{1}{54}x^3T_tT_{tt} - \frac{1}{54}y^3T_tT_{tt} + \frac{1}{9}u_{yy}y^2T_t^2 \\ &+ \frac{1}{9}u_{xx}x^2T_t^2 - XZ_1 + u_{yy}Y^2 + T^2u_{tt} + u_{xx}X^2 \\ &- \frac{1}{18}x^2XT_{tt} + 2u_{xy}YX - \frac{1}{6}x^2X_{tt}T + 2Tu_{yt}Y - \frac{1}{6}Ty^2Y_{tt} \\ &- \frac{1}{54}Ty^3T_{ttt} - \frac{1}{54}Tx^3T_{ttt} + 2Tu_{xt}X - \frac{1}{3}yZ_2T_t - \frac{1}{3}yY_tY \\ &- \frac{1}{9}y^2Y_tT_t + \frac{1}{3}u_yYT_t - \frac{1}{3}xT_tZ_1 + \frac{1}{9}u_yyT_t^2 + \frac{1}{9}u_xxT_t^2, \end{aligned} \tag{49}$$

where

$$\begin{aligned} &X_{1t} = XT^{-\frac{4}{3}}, \quad Y_{1t} = -YT^{-\frac{4}{3}}, \\ &Y_{2t} = Z_2T^{-\frac{2}{3}}, \quad Y_{3t} = Y^2T^{-\frac{5}{3}}, \end{aligned} \tag{50}$$

$$\begin{aligned} &Y_{4t} = Z_3T^{-1}, \quad Y_{5tt} = T^{-\frac{2}{3}}Z_2, \\ &Y_{6t} = T^{-\frac{4}{3}}YY_3, \quad Y_{7t} = XT^{-\frac{4}{3}}Y_9, \end{aligned} \tag{51}$$

$$\begin{aligned} &Y_{8t} = XT^{-\frac{4}{3}}Y_{10}, \quad Y_{9t} = T^{-\frac{2}{3}}Z_1, \\ &Y_{10t} = T^{-\frac{5}{3}}X^2. \end{aligned} \tag{52}$$

To determine the functions of K_1, K_2, and K_3, we substitute (32), (33), and (34) into (28) which yields the complicated equation

$$\begin{aligned} &J_{1,x} + J_{1,u}u_x + J_{1,u_x}u_{xx} + J_{1,u_y}u_{xy} + J_{1,u_t}u_{xt} + J_{1,u_{xx}}u_{xxx} \\ &+ J_{1,u_{xy}}u_{xxy} + J_{1,u_{xt}}u_{xxt} + J_{1,u_{yy}}u_{xyy} + J_{1,u_{yt}}u_{xyt} \\ &+ J_{1,u_{tt}}u_{xtt} + J_{2,y} + J_{2,u}u_y + J_{2,u_x}u_{xy} + J_{2,u_y}u_{yy} + J_{2,u_t}u_{yt} \\ &+ J_{2,u_{xx}}u_{xxy} + J_{2,u_{xy}}u_{xyy} + J_{2,u_{xt}}u_{xyt} + J_{2,u_{yy}}u_{yyy} \\ &+ J_{2,u_{yt}}u_{yyt} + J_{2,u_{tt}}u_{ytt} + \rho_t + \rho_uu_t + \rho_{u_x}u_{xt} + \rho_{u_y}u_{yt} \\ &+ \rho_{u_t}u_{tt} + \rho_{u_{xx}}u_{xxt} + \rho_{u_{xy}}u_{xyt} + \rho_{u_{xt}}u_{xtt} + \rho_{u_{yy}}u_{yyt} \\ &+ \rho_{u_{yt}}u_{ytt} + \rho_{u_{tt}}u_{ttt} = 0. \end{aligned} \tag{53}$$

To solve this equation, we start from the highest derivatives of u for K_1, K_2, and K_3 being $\{u_{xxx},\ u_{xxy},\ \cdots,\ u_{ttt}\}$ independent. Letting the coefficients of $\{u_{xxx},\ u_{xxy},\ \cdots,\ u_{ttt}\}$ be zero, we can get a more simplified equation. For example, the term of u_{ttt} in (53) is

$$27T^{\frac{4}{3}}K_{1,t_{12}}u_{ttt}, \tag{54}$$

where $K_{i,t_j^n}=\frac{\partial^j K_n}{\partial t_i^j}$. There is no non-trivial solution for $K_{1,t_{12}}\neq 0$, thus the only possible case is

$$K_{1,t_{12}}=0, \quad \text{i.e.,} \quad K_1\equiv K_1(t_1,\ t_2,\ \cdots,\ t_{11}). \tag{55}$$

Under (55), the coefficient of u_{ytt} is

$$27T^2(K_{2,t_{12}}+K_{1,t_{11}}), \tag{56}$$

which leads to the only possible solution

$$\begin{aligned} &K_2(t_1,\ t_2,\ t_3,\ \cdots,\ t_{12})=\\ &-t_{12}K_{1,t_{11}}+K_{21}(t_1,\ t_2,\ t_3,\ \cdots,\ t_{11}) \end{aligned} \tag{57}$$

with $K_{21}(t_1,\ t_2,\ t_3,\ \cdots,\ t_{11})$ being an undetermined function of the indicated variables.

Like the procedure to eliminate u_{ttt} and u_{ytt} and vanishing the terms of $u_{xxx}, u_{xxy}, \cdots, u_{yyy}$, we get $K_1,\ K_2,\ K_3$ expressed by 14 functions of $\{t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6\}$, which satisfy 14 linear equations. After solving out those equations we get the final solutions:

$$\begin{aligned} K_1=&\big[-\alpha_{1,t_5t_6}t_8+\alpha_{2,t_6}t_7+\alpha_{5,t_6}\big]t_{11}+\big[t_9\alpha_{1,t_5t_6}\\ &+(\alpha_{2,t_5}+\alpha_{1,t_4t_5}+\alpha_{6,t_5t_5})t_7+\alpha_{5,t_5}+\alpha_{1,t_1t_5}+t_4\alpha_{1,t_3t_5}\\ &+\alpha_{7,t_5t_5}\big]t_{10}+(\alpha_{3,t_6}-\alpha_{2,t_6}t_8)t_9+\big[-\alpha_{1,t_4t_5}-\alpha_{2,t_5}\\ &-\alpha_{6,t_5t_5}\big]t_8^2+\big[-t_5\alpha_{1,t_3t_5}+\alpha_{3,t_5}-t_4\alpha_{2,t_3}-\alpha_{2,t_1}-\alpha_{1,t_2t_5}\\ &+\alpha_{5,t_4}+\alpha_{8,t_5}\big]t_8+\big[(\alpha_{2,t_3}+\alpha_{6,t_3t_5}-\alpha_{10,t_4t_4})t_5+\alpha_{6,t_2t_5}\\ &+\alpha_{3,t_4}+\alpha_{2,t_2}+\alpha_{6,t_3}+\alpha_{6,t_1t_4}+t_4\alpha_{6,t_3t_4}-\alpha_{7,t_4t_4}+\alpha_{8,t_4}\\ &+\alpha_{11,t_4}\big]t_7+\big[-t_4\alpha_{10,t_3t_4}+\alpha_{10,t_3}+\alpha_{7,t_3t_5}-\alpha_{10,t_1t_4}\\ &+\alpha_{15,t_3}+\alpha_{5,t_3}\big]t_5+t_4^2\alpha_{6,t_3t_3}+\big[\alpha_{11,t_3}+\alpha_{8,t_3}-\alpha_{7,t_3t_4}\\ &+\alpha_{13,t_3}+\alpha_{3,t_3}+2\alpha_{6,t_1t_3}\big]t_4+\alpha_{11,t_1}+\alpha_{13,t_1}+\alpha_{5,t_2}\\ &+\alpha_{7,t_2t_5}+\alpha_{15,t_2}+\alpha_{10,t_2}+\alpha_{6,t_1t_1}+\alpha_{3,t_1}+\alpha_{8,t_1}\\ &-\alpha_{7,t_1t_4}+\alpha_{17}, \end{aligned} \tag{58}$$

$$\begin{aligned} K_2=&\big[\alpha_{1,t_5t_6}t_8-\alpha_{2,t_6}t_7-\alpha_{5,t_6}\big]t_{12}+\big[-t_9\alpha_{1,t_5t_6}\\ &+(-\alpha_{1,t_4t_5}-\alpha_{2,t_5}-\alpha_{6,t_5t_5})t_7-\alpha_{7,t_5t_5}-\alpha_{5,t_5}-\alpha_{1,t_1t_5}\\ &-t_4\alpha_{1,t_3t_5}\big]t_{11}+t_9^2\alpha_{2,t_6}+\big[t_8(\alpha_{1,t_4t_5}+\alpha_{2,t_5}+\alpha_{6,t_5t_5})\\ &-t_6\alpha_{1,t_3t_6}+(\alpha_{2,t_3}+\alpha_{6,t_3t_5})t_4-\alpha_{5,t_4}+\alpha_{4,t_6}+\alpha_{2,t_1}\\ &+\alpha_{6,t_1t_5}+\alpha_{9,t_6}-\alpha_{1,t_3}-\alpha_{7,t_4t_5})t_9+\alpha_{4,t_5}t_8+\big[(\alpha_{10,t_4t_4}\\ &-\alpha_{2,t_3}-\alpha_{6,t_3t_5}-\alpha_{1,t_3t_4})t_6+\alpha_{4,t_4}+\alpha_{9,t_4}+\alpha_{12,t_4}\big]t_7\\ &+\big[(-\alpha_{1,t_3t_3}+\alpha_{10,t_3t_4})t_4-\alpha_{1,t_1t_3}-\alpha_{7,t_3t_5}+\alpha_{10,t_1t_4}\\ &-\alpha_{10,t_3}-\alpha_{15,t_3}-\alpha_{5,t_3}\big]t_6+\big[\alpha_{12,t_3}+\alpha_{14,t_3}\\ &+\alpha_{9,t_3}+\alpha_{4,t_3}\big]t_4+\alpha_{9,t_1}+\alpha_{4,t_1}+\alpha_{12,t_1}+\alpha_{14,t_1}\\ &-\alpha_{16,t_1}+\alpha_{18}, \end{aligned} \tag{59}$$

$$\begin{aligned} K_3=&\big[-\alpha_{1,t_5t_6}t_{10}+\alpha_{2,t_6}t_8-\alpha_{3,t_6}\big]t_{12}+t_{11}^2\alpha_{1,t_5t_6}\\ &+\big[-\alpha_{2,t_6}t_9+t_6\alpha_{1,t_3t_6}-\alpha_{4,t_6}-\alpha_{6,t_1t_5}+\alpha_{1,t_3}+\alpha_{7,t_4t_5}\\ &-t_4\alpha_{6,t_3t_5}-\alpha_{9,t_6}+t_5\alpha_{1,t_3t_5}-\alpha_{3,t_5}+\alpha_{1,t_2t_5}-\alpha_{8,t_5}\\ &+t_8(\alpha_{1,t_4t_5}+\alpha_{2,t_5}+\alpha_{6,t_5t_5})\big]t_{11}+\big[(-\alpha_{1,t_4t_5}-\alpha_{2,t_5}\\ &-\alpha_{6,t_5t_5})t_9-\alpha_{4,t_5}\big]t_{10}+\big[(-\alpha_{2,t_3}-\alpha_{6,t_3t_5}+\alpha_{10,t_4t_4})t_5\\ &-\alpha_{6,t_2t_5}-\alpha_{3,t_4}-\alpha_{2,t_2}-\alpha_{6,t_3}-\alpha_{6,t_1t_4}-t_4\alpha_{6,t_3t_4}\\ &+\alpha_{7,t_4t_4}-\alpha_{8,t_4}-\alpha_{11,t_4}\big]t_9+\big[(-\alpha_{10,t_4t_4}+\alpha_{2,t_3}+\alpha_{6,t_3t_5}\\ &+\alpha_{1,t_3t_4})t_6-\alpha_{4,t_4}-\alpha_{9,t_4}-\alpha_{12,t_4}\big]t_8+\big[t_5\alpha_{1,t_3t_3}\\ &-t_4\alpha_{6,t_3t_3}-\alpha_{11,t_3}+\alpha_{1,t_2t_3}+\alpha_{7,t_3t_4}-\alpha_{8,t_3}-\alpha_{3,t_3}\\ &-\alpha_{10,t_2t_4}-\alpha_{13,t_3}-\alpha_{6,t_1t_3}\big]t_6+\big[-\alpha_{9,t_3}-\alpha_{4,t_3}-\alpha_{12,t_3}\\ &-\alpha_{14,t_3}\big]t_5-\alpha_{4,t_2}-\alpha_{12,t_2}-\alpha_{14,t_2}+\alpha_{16,t_2}-\alpha_{9,t_2} \end{aligned} \tag{60}$$

with $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5$, being arbitrary functions of $\{t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6\}$, $\alpha_6,\ \alpha_7,\ \alpha_8$ being arbitrary functions of $\{t_1,\ t_2,\ t_3,\ t_4,\ t_5\}$, α_9 being an arbitrary function of $\{t_1,\ t_2,\ t_3,\ t_4,\ t_6\}$, $\alpha_{10},\ \alpha_{11},\ \alpha_{12}$ being arbitrary functions of $\{t_1,\ t_2,\ t_3,\ t_4\}$, $\alpha_{13},\ \alpha_{14},\ \alpha_{15}$ being arbitrary functions of $\{t_1,\ t_2,\ t_3\}$, $\alpha_{16},\ \alpha_{17}$ being arbitrary functions of $\{t_1,\ t_2\}$, and α_{18} being a function of t_1.

Substituting (58)–(60) into (32)–(34), we can obtain the conservation laws relating to the symmetry (10) associated with the Lie-Bäcklund generator X_0. We have verified that the conserved vector $(J_1,\ J_2,\ \rho)$ really satisfies (28).

4. Summary and Discussion

In this paper, by applying the classical Lie method, we get the symmetry group of the potential Nizhnik-Novikov-Veselow equation (PNNVE) and prove it constituting the Kac-Moody-Virasoro (KMV) symmetry algebra. We generate the conservation laws of the PNNVE related to the infinite dimensional KMV symmetry group by use of the Lie-Bäcklund generator up to the second-order group invariants. The existence of arbitrary functions of the group invariants proves that the PNNVE has infinitely many conservation laws which are connected with the general Lie point symmetry (10). Though the symmetries and conservation laws are obtained from the PNNVE, we find that the conservation laws are only dependent on the symmetry which may be possesses many equations.

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